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Stability of Volterra Integrodifferential Systems

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The purpose of this paper is to use a decomposition–aggregation method and Liapunov functionals to discuss the stability of large scale systems described by liner Volterra integrodifferential equations.

1. INTRODUCTION

The stability of Volterra integrodifferential equations has been discussed by many authors [1–6]. In this paper we consider a system of integrodifferential equations of the form

$$\dot{x} = Ax(t) + \int_0^t C(t, s)x(s)\,ds, \tag{1}$$

where A is an $n \times n$ constant matrix, $C(t, s)$ is an $n \times n$ matrix continuous for $0 \leqslant s \leqslant t < \infty$.

Suppose system (1) can be decomposed as

$$\dot{x}_i = A_{ii}x_i(t) + \sum_{\substack{j=1\\ j\neq i}}^{m} A_{ij}x_j(t) + \int_0^t C_{ii}(t, s)x_i(s)\,ds + \sum_{\substack{j=1\\ j\neq i}}^{m} \int_0^t C_{ij}(t, s)x_j(s)\,ds, \tag{2}$$

where $i = 1, 2, ..., m$, $x_i \in R^{n_i}$, $\sum_{i=1}^{m} n_i = n$, A_{ij} is an $n_i \times n_j$ constant matrix, $C_{ij}(t, s)$ is an $n_i \times n_j$ matrix continuous for $0 \leqslant s \leqslant t < \infty$, and $\int_t^{+\infty} |C_{ij}(u, t)|\,du$ is continuous for $0 \leqslant s \leqslant t < \infty$, $i, j = 1, 2, ..., m$.

We also consider the ith isolated subsystem of system (2),

$$\dot{x}_i = A_{ii}x_i(t) + \int_0^t C_{ii}(t, s)x_i(s)\,ds. \tag{3}$$

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The purpose of this paper is to discuss the stability of system (1) with decomposition (2) via construction of Liapunov functionals for subsystems (3) and analyze its interconnecting structure.

2. Stability

In characterizing the stability properties of the isolated subsystem (3), it will be convenient to use the following additional nomenclature.

DEFINITION 1. Isolated sybsystem (3) possesses property A if

(i) there exists an $n_i \times n_i$ positive definite symmetric matrix B_{ii} such that

$$A_{ii}^{\mathrm{T}} B_{ii} + B_{ii} A_{ii} = -I_{ii}, \tag{4}$$

where I_{ii} is an $n_i \times n_i$ identity matrix;

(ii) there exists a constant $M_i > 0$ such that

$$|B_i| \left(\int_0^t |C_{ii}(t, s)|\, ds + \int_t^{+\infty} |C_{ii}(u, s)|\, du \right) \leqslant M_i < 1. \tag{5}$$

We consider the functional

$$V_i(t, x_i(\cdot)) = x_i^{\mathrm{T}} B_{ii} x_i + B_{ii} \int_0^t \int_t^{+\infty} |C_{ii}(u, s)|\, du\, x_i^{\mathrm{T}}(s) x_i(s)\, ds. \tag{6}$$

If subsystem (3) possesses property A, following Burton [2], we know that the zero solution of (3) is asymptotically stable.

THEOREM 1. *If the following conditions are satisfied,*

(i) *each isolated subsystem* (3) *possesses property* A;

(ii) *there exists a constant* $N_i > 0$ *such that*

$$|B_{ii}| \sum_{\substack{j=1 \\ j \neq i}}^{m} \int_0^t |C_{ij}(t, s)|\, ds + \sum_{\substack{j=1 \\ j \neq i}}^{m} |B_{jj}| \int_t^{+\infty} |C_{ji}(u, s)|\, du \leqslant N_i;$$

(iii) *the* $m \times m$ *text* $S = (s_{ij})$ *specified by*

$$s_{ij} = \begin{cases} 1 - M_i - N_i, & i = j \\ -2\, |B_{ii} A_{ij}|, & i \neq j \end{cases} \tag{7}$$

has positive successive principal minors, then the zero solution of (1) *with decomposition* (2) *is asymptotically stable.*

Proof. We choose a Liapunov functional

$$V(t, x(\cdot)) = \sum_{i=1}^{m} V_i(t, x_i(\cdot)) + \sum_{i=1}^{m} \int_0^t \int_t^{+\infty} \sum_{\substack{j=1 \\ j \neq i}}^{m} |B_{ii}| \times |C_{ij}(u, s)| \, du \, x_j^{\mathrm{T}}(s) x_j(s) \, ds, \tag{8}$$

where $V_i(t, x_i(\cdot))$ is defined by (6). Along the solutions of system (1) we have

$$\begin{aligned}
\dot{V}_{(1)}(t, x(\cdot)) = & \sum_{i=1}^{m} x_i^{\mathrm{T}} B_{ii} \Bigg(A_{ii} x_i + \sum_{\substack{j=1 \\ j \neq i}}^{m} A_{ij} x_j + \int_0^t C_{ii}(t, s) x_i(s) \, ds \\
& + \sum_{\substack{j=1 \\ j \neq i}}^{m} \int_0^t C_{ij}(t, s) x_j(s) \, ds \Bigg) \\
& + \sum_{i=1}^{m} \Bigg(A_{ii} x_i + \sum_{\substack{j=1 \\ j \neq i}}^{m} A_{ij} x_j + \int_0^t C_{ii}(t, s) x_i(s) \, ds \\
& + \sum_{\substack{j=1 \\ j \neq i}}^{m} \int_0^t C_{ij}(t, s) x_j(s) \, ds \Bigg)^{\mathrm{T}} B_{ii} x_i \\
& + \sum_{i=1}^{m} |B_{ii}| \int_t^{+\infty} |C_{ii}(u, t)| \, du \, x_i^{\mathrm{T}} x_i \\
& - \sum_{i=1}^{m} |B_{ii}| \int_t^{+\infty} |C_{ii}(t, s)| \, x_i^{\mathrm{T}}(s) x_i(s) \, ds \\
& + \sum_{i=1}^{m} \int_t^{+\infty} \sum_{\substack{j=1 \\ j \neq i}}^{m} |B_{ii}| \, |C_{ij}(u, s)| \, du \, x_j^{\mathrm{T}} x_j \\
& - \sum_{i=1}^{m} \int_0^t \sum_{\substack{j=1 \\ j \neq i}}^{m} |B_{ii}| \, |C_{ij}(t, s)| \, x_j^{\mathrm{T}}(s) x_j(s) \, ds \\
= & - \sum_{i=1}^{m} x_i^{\mathrm{T}} x_i + 2 \sum_{i=1}^{m} x_i^{\mathrm{T}} B_{ii} \sum_{\substack{j=1 \\ j \neq i}}^{m} A_{ij} x_j \\
& + 2 \sum_{i=1}^{m} x_i^{\mathrm{T}} B_{ii} \int_0^t C_{ii}(t, s) x_i(s) \, ds \\
& + 2 \sum_{i=1}^{m} x_i^{\mathrm{T}} B_{ii} \sum_{\substack{j=1 \\ j \neq i}}^{m} \int_0^t C_{ij}(t, s) x_j(s) \, ds
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{m} |B_{ii}| \int_{t}^{+\infty} |C_{ii}(u, t)|\, du\, x_i^{\mathrm{T}} x_i \\
&-\sum_{i=1}^{m} |B_{ii}| \int_{0}^{t} |C_{ii}(t, s)|\, x_i^{\mathrm{T}}(s) x_i(s)\, ds \\
&+\sum_{i=1}^{m} \int_{t}^{+\infty} \sum_{\substack{j=1\\ j\neq i}}^{m} |B_{ii}|\, |C_{ij}(u, t)|\, du\, x_j^{\mathrm{T}} x_j \\
&-\sum_{i=1}^{m} \int_{0}^{t} \sum_{\substack{j=1\\ j\neq i}}^{m} |B_{ii}|\, |C_{ij}(t, s)|\, x_j^{\mathrm{T}}(s) x_j(s)\, ds \\
= &-\sum_{i=1}^{m} \Bigg(1-|B_{ii}| \int_{0}^{t} |C_{ii}(t, s)|\, ds \\
&-|B_{ii}| \sum_{\substack{j=1\\ j\neq i}}^{m} \int_{0}^{t} |C_{ij}(t, s)|\, ds \\
&-|B_{ii}| \int_{t}^{+\infty} |C_{ii}(u, s)|\, du \\
&-\sum_{\substack{j=1\\ j\neq i}}^{m} |B_{jj}| \int_{t}^{+\infty} |C_{ij}(u, s)|\, du\Bigg) x_i^{\mathrm{T}} x_i \\
&+2 \sum_{\substack{i,j=1\\ j\neq i}}^{m} |B_{ii} A_{ij}|\, |x_i|\, |x_j| \\
\leqslant &-\sum_{i=1}^{m} (1-M_i-N_i) x_i^{\mathrm{T}} x_i + 2 \sum_{\substack{i,j=1\\ j\neq i}}^{m} |B_{ii} A_{ij}|\, |x_i|\, |x_j| \\
= &-W^{T} S W,
\end{aligned}
$$

where $W=(|x_1|, |x_2|, \ldots, |x_m|)^{\mathrm{T}}$. S is given by hypothesis (iii). From hypothesis (iii) we know that $\dot{V}_{(1)}(t, x(\cdot))$ is negative definite, then the zero solution of system (2) is asymptotically stable, and this completes the proof.

3. Corollary

The special form of system (1) is the convolution system

$$\dot{x} = Ax(t) + \int_{0}^{t} C(t-s) x(s)\, ds, \tag{9}$$

where A is an $n \times n$ constant matrix, $C(t)$ is an $n \times n$ matrix continuous for $0 \leqslant t < \infty$. Suppose system (9) can be decomposed as

$$\dot{x}_i = A_{ii} x_i(t) + \sum_{\substack{j=1 \\ j \neq i}}^{m} A_{ij} x_j(t) + \int_0^t C_{ii}(t-s) x_i(s)\, ds$$
$$+ \sum_{\substack{j=1 \\ j \neq i}}^{m} \int_0^t C_{ij}(t-s) x_j(s)\, ds, \tag{10}$$

where A_{ij} is an $n_i + n_j$ constant matrix, $C_{ij}(t)$ is an $n_i \times n_j$ matrix continuous for $0 \leqslant t < \infty$.

The isolated subsystem of system (9) has the form

$$\dot{x}_i = A_{ii} x_i(t) + \int_0^t C_{ii}(t-s) x_i(s)\, ds. \tag{11}$$

DEFINITION 2. Isolated subsystem (11) possesses property B if

(i) condition (i) of Definition 1 holds;

(ii) there exists a constant $M_i > 0$ such that

$$2\, |B_{ii}| \int_t^{+\infty} |C_{ii}(s)|\, ds \leqslant M_i < 1.$$

THEOREM 2. *If the following conditions are satisfied,*

(i) *each isolated subsystem* (11) *possesses property* B;

(ii) *there exists a constant* $N_i > 0$ *such that*

$$|B_{ii}| \sum_{\substack{j=1 \\ j \neq i}}^{m} \int_0^t |C_{ij}(s)|\, ds + \sum_{\substack{j=1 \\ j \neq i}}^{m} |B_{jj}| \int_t^{+\infty} C_{ji}(u)\, du \leqslant N_i;$$

(iii) *condition* (iii) *of Theorem* 1 *holds,*

then the zero solution of system (9) *with decomposition* (10) *is asymptotically stable.*

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REFERENCES

1. T. A. BURTON, Stability theory for Volterra equations, *J. Differential Equations* **32** (1979), 101–118.
2. T. A. BURTON, "Volterra Integral and Differential Equations," Academic Press, New York, 1983.
3. T. A. BURTON AND W. E. MAHFOUD, Stability criteria for Volterra equations, *Trans. Amer. Math. Soc.* **279** (1983), 143–174.
4. F. BRAUER, Asymptotic stability of a class of integrodifferential equations, *J. Differential Equations* **28** (1978), 180–188.
5. A. N. MICHEL AND R. K. MILLER, "Qualitative Analysis of Large Scale Dynamic Systems," Academic Press, New York, 1977.
6. R. K. MILLER, Asymptotic stability properties of linear Volterra integrodifferential equations, *J. Differential Equations* **10** (1972), 485–506.